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AUTHOR Bar-On, Ehud; Or-Bach, Rachel
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ABSTRACT

The development of an instructional model for teaching formal mathematical concepts (probability concepts) to disadvantaged high school students through computer programming and some results from a field test are described in this document. The instructional model takes into account both learner characteristics (cognitive, affective, and instrumental abilities) and subject matter requirements. The learner model is based on C. Frankenstein's theory about characteristic faulty patterns of thought among disadvantaged adolescents, namely, difficulties in abstract thinking, irrational thinking, and computational difficulties. The programming approach, which is concrete and procedural, provides a method of abstraction, generalization, and formalization of probability concepts and helps learners assimilate probabilistic concepts into their existing cognitive schemes. Following discussion of the model, examples of the process of detection, identification, and treatment of faulty patterns of thinking, as displayed in a pilot test of the approach, are presented. (RDN)

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PROGRAMMING MATHEMATICS: A NEW APPROACH IN TEACHING THE DISADVANTAGED

By

Ehud Bar-On and Rachel Or-Bach

Department of Education in Technology and Science

Technion - Israel Institute of Technology, Haifa 32000, Israel

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ABSTRACT

An instructional model for teaching formal mathematical concepts and operations to the disadvantaged high-school students is presented. Computer programming is used to express in a formal way experiments in probability and statistics. The instructional model has been constructed by considering both the learner cognitive, affective and instrumental abilities and the requirements of the mathematical subject matter. The learner model is based on Frankenstein's Theory about the characteristics of the disadvantaged adolescent. In this theory Frankenstein specified faulty patterns of thinking which are characteristic of the disadvantaged, together with general guidelines for their rehabilitation. The faulty patterns were classified in this study as difficulties in abstract thinking, irrational thinking (including emotional and motivational aspects) and computational difficulties. The formal and intuitive mathematical knowledge needed for understanding of basic probability concepts has been analyzed, and some basic concepts and experiments were chosen. The programming approach provided a method of abstraction, generalization and formalization of those concepts experimental results (e.g. random variables, sample space, equally probable events, uniform -gaussian -binomial distributions, histograms, central tendency and dispersion measures, etc.). This approach, which is more concrete and procedural, assisted in assimilating probabilistic concepts into the existing cognitive schemes of the learner. The concrete and procedural schemata could be replaced latter by a more formal schemata. Material for self-instruction was developed and a pilot study was made with disadvantaged students in vocational high-schools. Examples of the programming approach as well as examples of detection, identification and treatment of faulty patterns are presented. The suggested instructional model is expected to assist both less abled and regular students in the formalization

of mathematical experience.

PROGRAMMING MATHEMATICS: A NEW APPROACH IN TEACHING THE DISADVANTAGED

"It is remarkable that a science which began with consideration of games of chance should have become the most important object of human knowledge. ... The most important questions of life are, for the most part, really only problems of probability."

(Laplace, 1812)

Introduction

Teaching mathematics to disadvantaged junior-high school and high-school students is always problematic. The great difficulties they find in dealing with formalization, abstract concepts and abstract operations - which are the essence of mathematics - limits teaching to technical aspects of computation. This limitation leaves most of the more interesting and important fields in mathematics, like probability, out of reach of this population. Boring topics and tedious calculations, together with a history of failure in mathematics in previous years, make mathematic lessons an unpleasant experience both for the student and the teacher. These are reasons for searching for a more appropriate instructional model for teaching mathematics to the disadvantaged.

For developing an instructional model there is a need for a general model which specifies the relationships among learners characteristics (cognitive,

affective and instrumental), mathematical structure of the subject matter and the appropriate instructional methods. Such a general model could be used to derive a particular instructional model for subsets of learner population (e.g. disadvantaged adolescents) and subsets of the mathematical subject matter (e.g. probability). Lesh and Landau (1983) suggested, for the designing of better instructional experiences (in situations in which empirical evidence is not available), to generate theories for predicting the appropriate sequencing of ideas, tasks and instructional models. Such theories should specify a method for bridging the gap between the students' conceptual structures associated with the mathematical ideas (e.g. probability) and the formal mathematical structures that characterize them.

The purpose of this article is to suggest an instructional model for teaching formal mathematical concepts to students who have difficulties in dealing with mathematical abstractions and formalizations. In the construction of the teaching model both learner characteristics and subject matter requirements were considered. Frankenstein's Theory (1970,1979) about "rehabilitation of impaired intelligence" provided the basis for the learner characteristics. This theory helps to specify the expected cognitive, affective and instrumental (computational skills), characteristics of the disadvantaged student. Frankenstein's theory have been chosen because it provides a comprehensive model of the disadvantaged adolescent, together with an adequate teaching guidelines for this population. The disadvantaged student has been selected to provide better understanding of the mathematical difficulties, in the same way that many models of clinical psychology and medicine started from investigating the abnormal. Although the present instructional model was developed for the disadvantaged, many of the principles derived can be applied

in teaching probability to the more abled students. The disadvantaged are just the extreme end of a continuum, along which students could be placed according to their mathematical abilities. Basic concepts of probability have been chosen because of the important role of probability in everyday life. Probability have been chosen , also, because of its highly motivational nature. But, most of all, because it calls for critical thought. As Freudental pointed out: "In no mathematical domain is blind faith in techniques more often denounced than in probability ; in no domain is critical thought more often required" (Freudental,1970,p.167).

The resulting method employs self instruction material and microcomputer programming in teaching of basic concepts of probability to dicadvantaged high-school students. Self - instruction have been chosen in order to give the teacher the opportunity to treat properly the faulty patterns of thinking on an individual basis of interaction. The instructional - related model which will be developed in this article could be presented schematically as follows:

GENERAL INSTRUCTIONAL MODEL	A MODEL FOR TEACHING PROBABILITY TO DISADVANTAGED
Student's characteristics a) Cognitive - Abstraction formalization & generalization b) Affective - motivation, locus of control. c) Instrumental - Computational skills.	Disadvantaged adolescent a) Concrete thinking, inadequate differentiation b) Irrational thinking, irresponsibility, low motivation c) Univariate performance
↓	↓
Instructional model a) guidelines b) Technical aspects	Programming approach a)Formalization Through programming b) Self-Instruction + tutoring, computer experimentation
↑	↑
Subject matter a) Structure b) tasks	Probability concepts a) Set theory concepts & operations, probability axioms b) Experiments, data collection, tabulation, graphical display

The structure of this article follows the three blocks in our model. It starts with a brief review of the relevant literature for each block. Then, the relationships between the top block (learner) and the bottom block (subject matter) are discussed. The characteristics of the impaired intelligence, according to Frankenstein's theory, are described along with examples of faulty pattern of thinking and their manifestation in probabilistic thinking. Consideration of the learner and the subject matter results in the middle block, which points out the guidelines for designing an instructional unit for rehabilitative teaching. The method was especially designed to meet the difficulties imposed both by the population (first block) and the mathematical subject (third block). The last section presents application of the suggeste

instructional model. Examples of formalization through programming as it is employed in detection, identification and treatment of the mentioned faulty patterns are provided. Finally, some preliminary experimental results of applying the suggested method are discussed.

Literature review

Three topics will be reviewed corresponding to the three blocks in our schematic diagram: learning disabilities in mathematics, probabilistic thinking and programming approach in teaching mathematics.

a) Learning disabilities in mathematics - Ginsburg (1977) developed a conceptual scheme which consists of three systems of mathematical knowledge. This view describes two kinds of informal mathematical knowledge and one which is formal. "System 1" - is the intuitive mathematical concepts and techniques developed usually outside the formal school setting. It is informal and independent on culture. ""System 2" - is also informal but it is cultural dependent since it depends on culture transmission (by adults, Television, books, etc.). "System 3" - is both formal and cultural dependent. Allardice and Ginsburg (1983) in their review said that learning disabilities in mathematics have been attributed to different factors by different researchers. The main sources of mathematical difficulties which have been mentioned in the literature were:

   a) General deficiency - generally deficient intellectual functioning leading to poor achievements in mathematics (e.g.- genetic, neurological)

   b) Lack of intuitive concepts (e.g.- system 1: social deprivation).

   c) Lack of basic technical (computational) skills (e.g.- system 2: cultural deprivation).

d) Lack of knowledge of mathematical concepts and rules (e.g.- system 3: poor schooling).

e) Lack of skills of organization of mathematical ideas (metacognition)

f) Non-cognitive factors (e.g.- motivation, self-esteem, social context, etc.).

Russel and Ginsburg (1981), used Ginsburg's distinction between informal and formal systems of mathematical knowledge. they investigated the question whether elementary school mathematics failure results from retardation in informal knowledge (systems 1 and 2), or related to cognitive processes directly involved in formal mathematics. Their conclusion was that children with mathematical difficulties possess reasonably adequate informal concepts and strategies, i.e. - there is no developmental lag in this area. They suggested that the poor knowledge is related to failure in metacognitive processes. These results support Frankenstein's Theory of the disadvantaged adolescent. Frankenstein argued that the disadvantaged does not need more facts to memorize but needs the ability to organize his knowledge in a more generalized terms and structures. This organization depends on the abilities of abstraction, formalization and generalization.

b) Probabilistic thinking - Since we are interested in probabilistic thinking a brief review of the relevant research is in order. Recently, teaching probability as a mathematical topic, enjoys growing popularity as could be seen from the vast number of publications (e.g NCTM yearbook, 1981 is devoted to probability teaching). Genetic psychology has concerned itself with probability concepts like the concepts of chance, estimations of odds, and combinatorical capacity. The present study is particularly concerned with the influence of probability teaching on the understanding of abstract

probability concepts. Hawkins and Kapadia (1984) drew attention to the various definitions of probability: a-priori, frequentist, subjective and formal, and their relevance to pedagogical aspects. They recommended to blend the three approaches based on the first three definitions in order to provide children with an appropriate framework for studying formal probability. In teaching probability in school, one has to overcome the obstacle of deterministic thinking, which is emphasized in school mathematics and science curriculum. Stochastic behavior seems to contradict, what is usually stressed in school mathematics, that one counter example is sufficient to disprove a mathematical conjecture. Construction of a new conceptual system should use the (sometimes contradictory) pre-existing intuitive biases. Fishbein (1975) made probably the most significant contribution to our present understanding of intuitive sources of probabilistic thinking in children. He defined "intuition" as a global, synthetic and non-explicitly justified cognitive belief, which may or may not coincide with scientific knowledge. Fishbein (1975) pointed out that a proper curriculum of probability concepts should take into account the primary intuitive substrate, and find ways of developing new probabilistic intuitions which are compatible with probability theory predictions. Fishbein and Gazit (1984) found out that there is little information available about the effect of teaching probability on intuitive probabilistic judgment. They quoted a report by Lecoutre (1981) in which "university students who had received a prior formation in probability performed worse than those who had not received such instruction,...best performance was obtained by students with high practice in chance games". Her findings are consistent with Tversky and Kahneman results: studying probability theory does not immune against false predictions based on

intuitions (Kahneman, D. and Tversky, A.,1972). Fishbein and Gazit analyzed the effect of a teaching programme in probability devised for grades 5 - 7 pupils (10 - 13 years). They found that the programme improved probabilistic intuitions and had a beneficial effect on some intuitively based misconceptions. In this study, we are more interested in rehabilitation of formal thinking than in intuitions as an end in itself. That is, to develop in the disadvantaged student a differential and explicitly justified evaluation and prediction.

c) A programming approach to mathematical formalization - Finally, literature concerning the use of programming as a mean for assisting formal mathematical thinking will be reviewed. most of the many articles published on the use of computer programming in teaching mathematics (e.g. NCTM yearbook, 1984), emphasize the motivational aspects as well as the help in computation. Some authors of this yearbook suggested that if programming is to become a general strategy for mathematical education, then attention must be given to format, formality and methods by which it will be integrated into school mathematics (Camp, J. S. and Marchionini, G., 1984). Only few researchers addressed the more interesting question of the influence of programming on formal thinking in mathematics. Papert (1980) has suggested that programming in a "microworld" could provide a medium for the construction of formal operations out of the building blocks of logico - mathematical experience. Abelson and diSessa (1981) employed Papert's "Turtle Geometry" to formulate a mathematical system with axioms and theorems. The difference between conventional transformational geometry and the turtle geometry is that in the later, transformations manifest themselves as commands that result in changes in the states of concrete objects (lines on the screen and the turtle orientation angle). Knuth (1974)

suggested another aspect of the programming benefits: "The attempt to formalize things as algorithms leads to a much deeper understanding than if we simply try to comprehend things in the traditional way" (Knuth D. E.,1974 pp. 327). Forsythe claimed that: "The automatic computer really forces that precision of thinking which is alleged to be a product of any study of mathematics" (Forsythe G. E.,1959, cf. Knuth, 1974). The most important property of the programming approach, for the disadvantaged is that it could help in bridging the gap between the concrete and the abstract concepts and operations, and between the intuitive and the formal mathematical structures.

Learner faulty patterns and probabilistic misconceptions

According to Frankenstein's theory, the intellectual impairement is brought about by the externalizing conditions of poverty, neglect (social deprivation), cultural values ambiguity or rigid dependence on a rigid tradition (cultural deprivation). This intellectual impairement, also called "secondary retardation", can be characterized, according to Frankenstein's theory, by faulty patterns of thinking. Five faulty patterns, which are expected to influence the ability of the disadvantaged to understand formal mathematics, are presented. We classified the faulty patterns into three categories, corresponding roughly to the "cognitive", "affective" and "instrumental" facets of thinking:

a) Abstract thinking :

1)Generalization

2)differentiation

b) Rational thinking :

3)Rationality

4)Responsibility

c) Computational skills:

5) Concurrent processing

The corresponding faulty patterns will be presented along with examples of the respective probabilistic misconceptions.

a - Abstract thinking

1) Generalization - Inability to deal with abstract concepts :

It is manifested by a tendency to prefer concretizations to abstractions, substituting examples for generalities and laws and using specific rather than general concepts. Examples for such concretizations are the inability to recognize a "tossing of a coin" as a special case of "an event space consisting of two equally probable events ". Another example is thinking that throwing any regular (or irregular) polyhedron will result in a six equally probable outcomes. These mistakes results from the inability to deal with the abstract mathematical concepts of "events", "event space" and "equally probable events" which are replaced by the concrete and familiar experience with coins and dice.

2) Differentiation - Inadequate differentiation :

The disadvantaged has problems to distinguish between similarities, to recognize nuances, to use analogies and to relate to an hypothetical situation. Inadequate differentiation is also manifested by a tendency to use dichotamizations as means of organizing the multitude of knowlable and learnable data. An example of inadequate differentiation is the inability to see the probability of occurrence of events as a continuoum ranging from "certain" (p=1.0) to "impossible" (p=0.0). The disadvantaged tend to use dichotomization : each event is either "certain" or "impossible". I.e. identifying the "highly probable" with "certainty" and "low probability of occurrence" with "nonexistence" (Falk et al.,1980)

b - Rational thinking

3) Irrational thinking :

Thinking is governed by uncontrolable associations, associations that are not relevant by content and are usually emotionally driven. Examples of irrational thinking are:

a) Preferences for a favorite colour influence students' predictions.

b) Interpretation of the objectively random as the

manifestation of the "will" of the object concerned.

c) A belief that random events can be controlled by the person who triggers the events.

d) Passive induction: predicting event's probability of occurrence on the basis of the immediately preceding occurrence, and not on the basis of a deductive scheme (e.g. a combinatory scheme).

4. Responsibility - Lack of responsibility for the acts of thinking and learning :

External locus of control, passive reliance on the authority of a teacher or a book. Belief in fate and arbitrariness, instead of being interested in casual relationships. Examples of external locus of control are beliefs in "good luck" and "bad luck" as influencing objective probabilities.

c -Concurrent processing

5. Concurrency - Inability to see different aspects simultaneously

Inability to see the different aspects of a problem, a task or a phenomena simultaneously. Examples of that faulty pattern of thinking are:

a) The inability to take into account, while calculating probabilities, both the total number of elements comprising the event space, and the number of the specific pertinent elements (comprehension of proportion).

b) The difficulties in understanding operations on sets, like union and

intersection. In probability theory it is manifested by the difficulty in comprehension of "joint events".

c) The inability to understand the idea of "conditional probability", i.e. the idea that the probability of one event is dependent on the probability of another event.

There is a major difference between Frankenstein's model of the disadvantaged adolescent and the more familiar model of the child in the pre-formal stages proposed by Piaget and Inhelder (195). It is important to understand the differences between the two models. Although the phenomenology is similar in many cases, the treatment called-for by the basic assumptions of the two models is entirely different and even contradictory. According to the common interpretation of Piaget's theory, the development of probabilistic reasoning is related to the development of operational schemes of the intellect. Children which are in the intuitive stage and the stage of concrete operations should be taught, according to this interpretation of Piaget theory, in a way which suits their intellectual stage capabilities. It is a common practice to treat the disadvantaged adolescent in the same way as younger children in a pre-formal stage (Minkovitz, A.,1976). Frankenstein's theory states that the disadvantaged adolescent should not be taught according to his demonstrated intellectual capabilities. Instead, he should be taught in a formal and abstract level in a specific way designed to rehabilitate his impaired intelligence. According to Frankenstein's theory, the secondary retardation is reversible if a suitable education takes place during adolescent years.

The characteristics of the faulty patterns of thinking mentioned above and the nature of the subject matter guided the development of the instructional material.

General Guidelines for the Instructional Model

The guidelines will be divided into those which are derived from the disadvantaged characteristics, those which are derived from the mathematical subject matter (probability) and those which take advantage of the special properties of the programming approach for teaching probability to the disadvantaged. This section is concluded with an example which points out the differences between an approach using formal mathematics and the programming approach to the teaching of mathematics.

Guidelines derived from learner characteristics

Deductive approach

The learning takes place in a microcomputer laboratory. Each student is required to read definitions and explanations, to answer questions and to perform experiments according to working sheets. Experimentation using microcomputers as a mathematical laboratory is usually associated with "discovery learning". Mathematical discovery, as suggested by Polya is based on inductive reasoning ability. Polya (1965) recommended consecutive phases of: exploration, formalization and assimilation. According to discovery learning principles, students are expected to explore mathematical phenomena and discover rules by employing inductive reasoning. Frankenstein opinion is that this approach is not suitable for the disadvantaged cognitive abilities. He believes that the disadvantaged lacks the self-confidence needed to abandon

the safe ground of the familiar and concrete, in the search for the unknown and abstract. Free association together with emotionally driven conclusion and the tendency to give up after unsuccessful attempts, makes it very unlikely that the disadvantaged could discover by himself anything which is useful. Therefore, a deductive approach seems to be more appropriate for him. According to this approach, principles, rules and solutions are posed to the student beforehand and he is guided in finding possible examples of the explained laws, in recognizing the replaceability of one solution by another, and in finding the limits of the laws. The disadvantaged is unable to write his own programs because he is unable to deal with the abstract concepts (e.g. variables, counters, arrays) and operations (e.g. loops, control structures) of programming. Therefore, the student is provided with an example program and he is guided how to execute it, how to interpret the outcomes and how to modify it in some pertinent ways. In other words, the student is provided with the formal representation of the probabilistic experiment (in the form of a simple computer program). Commencing with the computer program, he is taught to identify and explore the relationship between this formal representation and the real world on one hand, and its relation to mathematical concepts on the other hand.

Neutralization

According to Frankenstein's theory, an important goal of rehabilitative instruction is to detect, identify and treat irrational thinking. This goal is accomplished by choosing topics which tend to be emotionally neutral. E.g. - too much emotional involvement in the games of chance and high motivation to win might interfere with rational thinking about the probability concepts.

This guideline will lead to demphasis the game-like nature of the experiment and to concentrate on its formal representation as a computer program, which is neutral emotionally.

Treatment of faulty patterns of thinking

The treatment of students errors is crucial to the rehabilitative instruction. One of the main goals of the teacher is to create situations in which students mistakes are elicited and treated. It is instructive to compare the treatment of errors in Frankenstein's theory to Brown and VanLehn (1980) "repair theory". Repair theory draws its name from the belief that when a student has unsuccessfully applied an algorithm to a given problem he will attempt a repair. Attempting to rigorously follow the impoverished algorithm might lead to an impasse . The student will often attempt to repair the "impasse" so he can continue, albeit in an erroneous way, thus creating an errors or "bugs". Brown and Van Lehn speak about "core procedures" or algorithms which represents the student's current knowledge. Both repair theory and Frankenstein's theory are relating to errors as symptoms of a deeper construct which can be only inferred. They differ in their assumptions about the deep constructs. Repair theory assumes that the core procedures are results of instruction, and systematic errors are products of incomplete or misguided learning. Therefore the treatment will be by suggesting a better procedure. According to Frankenstein's theory the errors suggest a faulty pattern of thinking, and therefore it is the faulty pattern that should be treated. This rehabilitative approach will be examplified later in our article.

Experimentation in probability

As was mentioned above, Fishbein and Gazit (1984), who suggested a teaching programme for teaching probability by experimentation, stressed the importance of developing probabilistic intuitions. We have seen that Russel and Ginsburg (1981) had found that disadvantaged children have not suffered from a deficiency in their intuitive or informal mathematical knowledge (systems 1 and 2), but only in their formal knowledge (system 3). The ability of the disadvantaged to acquire informal knowledge is believed to be a prerequisite for formal knowledge. In Piaget's view, the learner assimilates academic material into the already existing schemata: the child's informal knowledge serves as a kind of cognitive underpinning or scaffolding for formal learning. Although the development of probabilistic intuitions is not an end in itself in our method, intuitive understanding can pave the way for abstraction and generalization. Fishbein (1975) and Fishbein and Gazit (1984) argued that formal knowledge cannot change well embedded intuitions, but as intuitions are syntheses of individual experience they might be improved or modified by a directed involvement in preplanned activities. Probability is a suitable subject for conducting experiments and The microcomputer is an ideal tool for constructing a mathematical laboratory for this topic. Simulation of probabilistic experiments could be accomplished very conveniently on a microcomputer. However, it is only the understanding of the correspondence between the computer program representation and the real experiment on one hand, and probability concepts on the other hand, which serve the goals of rehabilitative teaching.

Guidelines derived from the combination of learner and subject matter

Procedural approach through programming

It is our conjecture that a programming approach could provide means of formalization of mathematical knowledge in a way which is more comprehensible to the less abled, than the usual mathematical formalization. Formalization is the process by which mathematics is adapted for machine processing. To program a computer to represent experiments in probability, one must know the vocabulary and the grammar rules of the computer language being used. Therefore, computer languages are formal languages and programming of a probabilistic experiment is a way of formalization the mathematical knowledge involved in the experiment. However, this kind of formalization has five properties which makes it especially suitable for this population:

a) Less abstract - In most computer languages, variables have specific values in specific points of time. The user can always stop the program and print their current values. The written or graphic output corresponds to those values. This property gives the filling of manipulating concrete mathematical objects rather than abstract symbols.

b) Discrete and finite - The representation in a computer language is always discrete. Continuous sets are sampled into discrete numeric values. The range of values is not infinite but finite, and often when dealing with integers, it is even finite and small.

c) procedural representation - Mathematical solutions are represented by a procedure for solution or an algorithm. The computer algorithm is deterministic: the outcome is completely determined by the algorithm and

the input.

d) Serial processing - Execution is done step by step, one step at a time. This property enables the student to follow complicated algorithms by following the simple small steps.

e) Automatic evaluation of expressions - The user could write complicated algebraic expressions without having to worry about their numeric evaluation.

Properties a,b and c helps in assisting the disadvantaged in developing his formal and abstract thinking. Property c is of great value in treating irrational and irresponsible patterns of thinking. Properties d and e are important to deal with the computational difficulties.

The derived guidelines for the instructional model will be to take advantage of this properties in the teaching method. Examples of the implementation of those guidelines in the programming approach will be brought in details in the next section.

An example comparing between formal approach and the programming approach

In order to clarify the guidelines for the suggested programming approach, we will conclude this section by pointing out the merits of the programming approach for the less abled. The formal language which is usually used in probability theory textbooks is mainly the language of formal set theory. The language consists of symbols specific to set theory, symbols of logic, symbols of operations etc. In order to illustrate the comparison, we shall bring an example of presenting a theorem in probability in both ways. The formal definition of probability will be given as a function of events on the subsets

of a sample description space of a random phenomenon:

Definition - Let S be a sample description space on the subsets of which a probability function P(.) has been defined. A probability function assigns to every event A on S a nonnegative real number, denoted by p(A), and called the probability of event A, such that the following axioms are true:

1. $p(A) \geq 0$ for every event A.

2. $p(S) = 1.0$

3. $p(A \cup B) = p(A) + p(B)$, if events $A, B \in S$ satisfy: $A \cap B = 0$.

Proofs of theorems are accomplished by using symbolic logic inference rules. This is an extremely unmotivated and abstract way to discuss probability which could not be understood by the disadvantaged. The programming approach provides procedural definitions of the experimental situations. For example, in one of the programs to be introduced in the next section, points are distributed randomly and uniformly on the computer screen. The 1200x1000 grid points of the screen (S - the "sample space") have equal probability - p(A) - of being chosen. Within this screen-grid representation, the first axiom is interpreted as the intuitive notion that there are no "holes" in the grid, i.e. every point has a nonnegative chance of being chosen. The second axiom is interpreted as the assertion that each generated point will be included within the limits of the screen. The third axiom is equivalent, in our interpretation model, to the assumption of the additivity of probabilities. The probability of a point to fall inside a region on the screen is linearly proportional to the proportion of its area to the total screen area. Using this understanding we could "derive" theorems expressed in a procedural way. For example, the theorem:

$$p(A) = 1 - p(\bar{A})$$

needs for its formal proof, the use of axiom 2 :

$$p(A) \cup p(\bar{A}) = p(S) = 1)$$

and axiom 3:

$$p(A) + p(\bar{A}) = 1 \; ; \; A \cap \bar{A} = 0$$

in the screen-grid model, if p(A) is the probability of a point to fall within the right half of the screen, and p(A) is the probability of falling outside of this region, then the computer instruction (in the BASIC language), which will count the frequency F of points falling into the first region is:

```
IF X <= 600 THEN LET F = F + 1
```

The fact that a point x in S is a member of set A for every x such that x is less or equal 600, is expressed as an instruction: If x is less-equal 600 then do something. The truth value of this sentence is checked for each x in its turn and not on the set of all x members of S. The complementary event is obvious: it is all x in S for which this condition does not hold. This way of presenting theorems is very different from the traditional theorem prooving, but it is comprehensible and yet formal and enables abstraction, generalization and differentiation of mathematical concepts.

Formalization of intuitions - A programming approach

This section is organized according to the faulty patterns of thinking which were mentioned before as characterizing the disadvantaged. For each of these patterns the programming approach is demonstrated. The order of presentation is determined by the computer programs involved. BASIC computer language has been used, because it is the most widely used language in schools, but any

computer language will do for this purpose.

1. The first faulty pattern that is tackled here concerns the difficulties in abstraction and generalization

Examples of formalization of probabilistic experiment in order to understand the concepts of "random variable", "outcome space":

a) Simulation of throwing a regular polyhedron or spinning of a top - the built-in BASIC function: RND(N), generates a random integer number which is uniformly distributed over the range 1 to N. In this simulation N is the number of faces of a regular polyhedron or top (which are numbered from 1 to N). The disadvantaged student has to formalize his intuitions of "randomness", "outcomes space" and "equally probable events". This formalization is achieved through programming in the following way:

INSERT FIG I NEAR HERE

The computer program simulates 30 possible outcomes of throwing a n-faces polyhedron or spinning a n-faces top. The importance of this formalization for the disadvantaged student is that he has to realize that the only crucial feature to be simulated is the number of possible outcomes (which corresponds to the number of faces). Therefore, two different concrete objects like a polyhedron and a top are represented in the same way, i.e. they are isomorphic. Formalization through programming enables the student to try out whether the results are

consistent with the expected outcomes. The random variable X is the formal representation of the abstract concept of a "random variable" and yet something which can be investigated by experiments.

b) It is very important in general, and particularly to the disadvantaged, to teach the same abstract concepts in entirely different contexts. The concepts of "randomness", "random variable", "outcomes space" and "equally probable events" can be demonstrated through another example. The example is based on a random distribution of points on the computer screen:

```
10 REM * STARRY NIGHT *
20 MODE 5 : REM * GRAPHIC MODE*
30 FOR I = 1 TO 3000
40    LET X = RND(1200)
50    LET Y = RND(1000)
60    MOVE X,Y : DRAW X,Y
70 NEXT I
```

INSERT FIG II NEAR HERE

The computer program above distributes 3000 bright points on the dark background of the screen creating an image of a "starry night". The "randomness" is demonstrated by the unpredictable position of the next point Since the values of both "random variables": X and Y, are drawn from a uniform distribution, the points are spread evenly on the screen. That is, on the average, equal number of points will fall into regions

which have equal areas. The "outcomes space" consists of 1200X1000 possible positions on the whole number grid representing the screen. The uniformity of the spreading of the points demonstrates the equal probability of the 1200X1000 outcomes. The same program could be used in a later stage to illustrate the independence of the two random variables X and Y. These two different examples are important because they act against the disadvantaged tendency to identify probability concepts with concrete objects like dice, coins, spinners etc. and concrete operations like tossing or spinninig.

2. The second faulty pattern which will be discussed is the disadvantaged inability to deal with two or more variables simultaneously. In our classification, this inability was interpreted as representing difficulty in performing intersection or union (or the corresponding "AND" or "OR" logic operators) on sets. This disability leads to difficulties in understanding joint events. In the formerly presented program - "STARRY NIGHT" - two random independent variables i.e X and Y determine the location of each "star" on the screen. The X random variable determines the horizontal position and the Y random variable determine the vertical position, but it is the combination of both values which determines the joint event: i.e. the location of a single point. The program instructions 40, 50 and 60 are an example of how programming may assist in understanding how two independent variables may act concurrently.

3. Another faulty pattern of thinking of the disadvantaged student is irrational thinking and involvement of emotions while analyzing data and making conclusions. This is particularly pronounced when dealing with the

rules of chance, which were always mysterious and emotionally loaded. Ancient Romans and Greecs use to sacrifice to Tyche, goddess of luck, holding the metaphysical position that god speaks to humans through the action of chance (Davis, P. J. and Hersh, R., 1980). Children who study the "Old Testament", learn how Israelites selected King Saul by casting lots (Book of Samuel 1, 10: 20-21). The implication of this story is that the results of casting of lots is an expression of the divine will. Tossing of coins is still a common device for decision making in the face of uncertainty. The programming approach helps to think in a rational way and helps to realize what factors influence the outcomes. The programming examples above intend to make clear that the influencing factors are the number of faces, in the first example, and the grid dimensions in the second one. It is instructive to add a new random variable C which determines the colour of the points in the "STARRY-NIGHT" program. It is clear from the resulting picture that the colour is just an epiphenomenon and does not influence the distribution of points.

Another manifestation of irrational thinking is the tendency to judge facts on the basis of the immediately preceding facts - "passive induction". An example of this fallacious reasoning, known also as "gamblers' fallacy" or "Monte-Carlo fallacy" is the belief that a run of heads in a coin-flipping, increases the likelihood that tails will come up the next turn. Experimenting with large numbers help to reveal the overall lawfulness instead of paying attention to the immediately preceding fact. That is how the "STARRY NIGHT" program helps to contradict "passive induction".

4. The fourth faulty pattern that will be dealt with is inadequate differentiation. The notion of statistical laws is very important for the

disadvantaged student who tends to a dichotomized way of thinking and tends to arrive at conclusions on the basis of one example. It is important that he will understand that even though it is impossible to know a specific outcome,it is possible to discover the lawfulness for a large number of experiments. The programming example deals with a directed random walk on a rectangular grid. In the following example 300 trials or experiments are performed. The random walk on the grid starts at the upper left corner of the screen. In each one of 10 nodes in the trajectory of the random walk there are 2 alternatives i.e "elementary events": either walking downward or to the right. In our example, both elementary events are equally probable (P=0.5). At the end of each possible trajectory there is an end-point. There are N+1 end-points i.e "joint events", all of which are equally distant from the starting point (the distance is measured in a "City-Block" space). Students tend to expect that all end-points will be reached an equal number of times. The following program draws the 300 trajectories and displays graphically the relative frequencies in which the end-points had been reached:

```
10 REM  * BI-DIRECTIONAL RANDOM WALK *
20   MODE 5
30   LET NT = 300 : LET N = 10 : REM * NT = NO.OF TRIALS; N = NO. OF NODES*
40   DIM H(N) : LET TX =1200 / N : LET TY = 1000 / N : LET P = 0.5
50   FOR J = 1 TO NT
60      LET IX = 0 : LET X = J/10 : LET Y = 1000+J/10 : MOVE X,Y
70      FOR I = 1 TO N
80         R =  RND (1)
90         IF R <  P THEN X = X + TX : LET IX = IX + 1
```

```
100         IF R => P THEN Y = Y - TY
110         DRAW  X,Y
120       NEXT I
130       H(IX) = H(IX) + 1
140       REM ****************** DRAWING THE FREQUENCY POLYGON **********
150       FOR L = 0 TO N - 1
160         IF L = 0 THEN  MOVE 0,H(0)
170         DRAW L*TX,H(L)*10
180       NEXT L
190     NEXT J
200 END
```

INSERT FIG. III NEAR HERE

The resulting frequency polygon of the joint events demonstrates the fact that not all the end-points have been reached with the same frequency. It might be very instructive to run the program first with NT=1 (NT=number of trials) and then with large NT. When NT is small the students have the feeling that since the end-point of a specific trajectory is unpredictable, nothing can be said about the probability of reaching a particular point. Our experience with the disadvantaged adolescents is consistent with Falk's experience with children (Falk et al.,1980). Probabilities which are less than 0.5 are considered as not conveying any information about the expected results. On the other hand, high probabilities are equated with certainty about the results. Runing the program with large NT helps to demonstrate that although single trials end-points cannot be predicted, the frequency

polygon shows a predictable pattern. It is also instructive to ask the students to change the value of P and to investigate its influence on the shape of the frequency polygon. A very important consequence of running this program is the refinement of the disadvantaged coarse differentiation. He learens that in spite of the low probability of the joint events represented by reaching extreme end-points they still happen, and in spite of the relativly high probability of the joint events represented by ending in the middle end-points they are not "sure".

5. The last characteristic of the disadvantaged student to be discussed is irresponsibility for acts of thinking and learning. This characteristic, as described by Frankenstein, is similar to the better known concept of external locus of control. It has been pointed out that persons with external locus of control (which is highly correlated with low need to achieve), tend to enjoy games of chance (McClelland, D. C., 1961). It is the same trait which creates high motivation in the disadvantaged population, which interfere with taking responsibility for thinking and learning. One should be cautious not to substitute the student's passive reliance on the teacher by a passive reliance on the computer. Therefore, we believe that it is necessary that the student will program the computer instead of using ready software. By typing in the instructions and changing the program, the disadvantaged feels active and responsible for the act of learning. Runing the "BI-DIRECTIONAL RANDOM WALK" program and changing the parameters each time help the disadvantaged to realize the causal relationship between the probability of an event and the expected result. Programming games of chance help to clarify that results depend on objective probabilities and not on "good" or "bad" luck.

Protocols examples from the field experiment

We shall conclude our theoretical article by some examples of the treatment of faulty patterns of thinking in the field. The suggested method of rehabilitative teaching of probability concepts was applied in a vocational high school with disadvantaged population. The students suffered from difficulties in simple arithmetics as well as very low motivation for studying mathematics. The students were quite enthusiastic about playing games of chance on the microcomputers. However, high motivation toward "mathematical activities" was not considered as being satisfactory. We were interested in rehbilitative teaching which relies heavily on the ability of the teacher to treat errors which represent faulty patterns of thinking. The instructional material, constructed according to the suggested method, provides an environment in which faulty pattern of thinking could be revealed and treated. But the teacher plays a central role in detecting the errors, identifying the faulty pattern, and treating them according to the suggested guidelines.

The process of detection, identification and treatment of faulty patterns will be illustrated with three examples from protocols. The three examples correspond to our classification of faulty patterns into three classes.
All three examples were collected when students worked on the "STARRY NIGHT" program

a) Error - Q.: Does the location of a new point on the screen depends on the location of previous points ? ("STARRY-NIGHT" program)

A.: Yes, because some of the locations are already occupied.

Detection : Inability to think on an abstract level.

Identification: The student knew that the location of the point is

determined by the values of the random variables X and Y. He also knew that X and Y are independent of previous values. Yet, when he saw the points on the screen, he was treating them as concrete objects. Two or more concrete objects cannot occupy the same space. Therefore the student concluded that the location of a new point depends on previous locations.

Treatment : The treatment of this error consists of pointing out the difference between concrete and abstract objects, and then going again through the above statements and reaching a correct conclusion. The screen grid (The programmed or symbolic model) is compared to a 6x6 rectangular grid, drawn on a plywood plate, with holes in the node-points. Two dice - red and blue - are rolled and pegs are inserted in the holes which coordinates correspond to the resulting red-blue ordered pair (The concrete model). The student was asked to compare the two models i.e to point out their similarities and differences. The student pointed out that both models are similar with respect to the way points on the grid are determined. They are different since "one cannot produce a cube with 1000 or 1200 faces.." This answer is an example of the student inability to make an abstraction of the concrete dice. Then he was asked to compare the two models with respect to the assumption of independence of successive trials. The student was guided to the conclusion that in both models, a situation in which occupied nodes have zero

probability of being chosen, contradict the assumption of independence. In the concrete model the student avoids inserting pegs in occupied holes. In doing so, one is violating the assumption of independence. In the programmed model, if an equivalent avoidance is desired, it should be seen from the program instructions. Since there is no such " IF ... THEN .." instruction in the "STARRY-NIGHT" program, the students former conclusion about need for vacant nodes is erroneous.

b) Error - Q: If you divide the screen into two halfs, choose one of them and run the program "STARRY-NIGHT" 100 times. how many times you expect to win (winning is defined as having more points in the half-screen chosen by the student) ?

A1: I'll probably loose. I am a looser.

A2: I don't have any chance of winning this game. The computer distributes the points, so he will fix it in a way that he will win.

Detection : Both answers represent irrational thinking.

Identification: The students knew that the points are randomly and uniformly distributed. They also knew that regions with equal area are expected to contain, on the average, equal number of points. They knew as well that the two halfs have equal chance of containing more than half of the points. However, when they interpreted the results in terms of winning and loosing they could not think rationally.

Treatment : The treatment was to neutralize the emotionally loaded terms of a "winner" and a "looser". The students were asked to compare the "win-lose" marking with an alternative marking (boolean variable) like "one-zero", "one-two", "yes-no", etc. It was emphasized that the computer does not do anything which does not appear as a program instruction. The computer is unable to "know" if he is "winning" or "loosing", let alone to fix the results. The students programmed the computer so they are responsible for anything done by the computer.

c) Error - Q.: If you draw a rectangular 10x10 grid on the screen, and count the points falling in each one of the columns and each one of the rows, you might expect equal number of points in each column or row. The instruction which generated the values of X is changed so the range is decreased from $1 < X < 1200$ to $1 < X < 600$. Draw the expected result on the screen and explain how do you expect this change, in X values, to effect the range of points in the rows ?

A.: If you cut the range of X by half, half of the columns and half of the rows will become empty. If you change the X values, you change the location of the points, so everything will be changed.

Detection : Inability to understand concurrent change in two or more variables.

Identification: The student knew that X and Y values are generated independently. He also knew that X value determines the horizontal location and Y value determines the vertical one. However, the fact that their combination determines the location of a single point, lead him to the wrong conclusion that a change in the range of X, will cause a similar change in the range of Y.

Treatment : The simplest way to show the student that he was wrong is to let him try it on the computer and discover for himself that changing the range of X, does not effect the range of the distribution of Y. Modifying instructions 40 and 50 in the "STARRY-NIGHT" program may help in forming the intuition of independence. However, in order to enhance understanding of this independence in the formal level, the student was asked to modify the program to produce an outcome which will match his prediction (i.e. "half of the rows and half of the columns will be empty"). In order to do so the student has to formulate his thought in a formal (programming) language. Limiting the points to the region where $1 < X < X1$ and $1 < Y < Y1$, could be generalized to the joint event which can be represented by the

rectangular region: X1 < X <X2 and Y1 < Y < Y2. Confining the points to this region can be achieved by either, generating X and Y values in the specified ranges, or by filtering the points. The filtering strategy has the advantage that the student has to use symbolic operators. The resulting instruction is a formalization of a joint event written in programming language. E.g., line 60 in the program "STARRY-NIGHT" could be replaced by this line (do not forget to assign values to X1,X2,Y1,Y2) :

```
60  IF (X>X1 OR X<X2) AND (Y>Y1 OR Y<Y2) THEN MOVE X,Y : DRAW X,Y
```

Conclusion

In this article, an attempt was made to construct an instructional model for teaching formal mathematical concepts through programming. The model construction was achieved by a combination of a "top-down" strategy (theory about the learner cognitive, affective and instrumental abilities), and a "bottom-up" strategy (task analysis of programming and probability). The learner model was based on the one proposed by Frankenstein for the disadvantaged adolescent. The choice of the disadvantaged was made both because we were interested in this population, and because the disadvantaged characteristics might be used to learn about the "normal" student. Many models in medicine, psychology and other sciences, started from the characteristics of the extreme case and the abnormal. In the disadvantaged , one can see cognitive, affective and instrumental disabilities which exist, to a certain degree, in most of the learners of mathematics. Our impression is, that the same model could be applied successfully to teaching mathematics to

ordinary high-school and junior high-school students. Preliminary experimental results indicated that instructional material written according to the suggested model could provide an appropriate environment for eliciting and treating faulty patterns of mathematical thinking. It is essential that the teacher will be familiar with the suggested method, so he could detect, identify and treat the faulty patterns of thinking. Our first results are encouraging. In the future we hope to refine and formalize our model, to enable its implementation as a computerized expert system. Such an expert system could replace the teacher in detection, identification and treatment of faulty patterns manifested through the students' mistakes.

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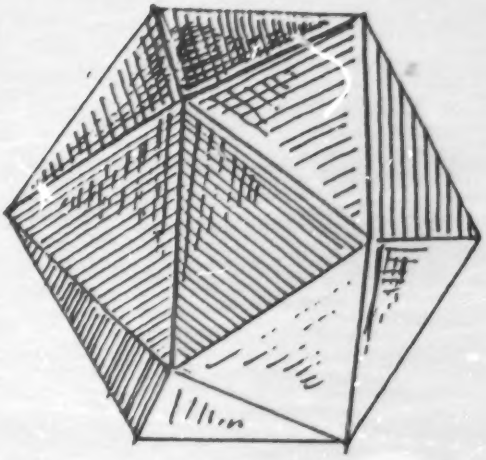
LEGENDS

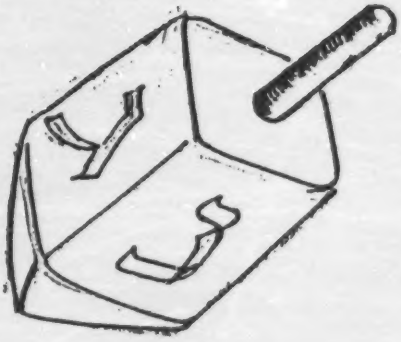
Fig. I: Computer program representation of a sample space with N equally probable events. The same formal representation could be used to represent throwing of dice, coins, regular polyhedrons or spinning of tops etc.

Fig. II: The program "STARRY NIGHT" distributes 3000 bright points or "stars". The points are randomly and uniformly distributed on the screen.

Fig. III: Bidirectional random walk on a rectangular grid. Starting point is in the upper left node. at each node there are equal probabilities ($p = 0.5$) of walking down or to the right. Each trial consists of a sequence of 10 steps. At the bottom of the screen the frequency polygon describes the frequency of reaching each one of the 11 end-points.



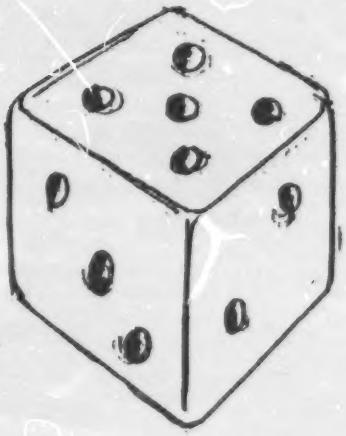
n = 20



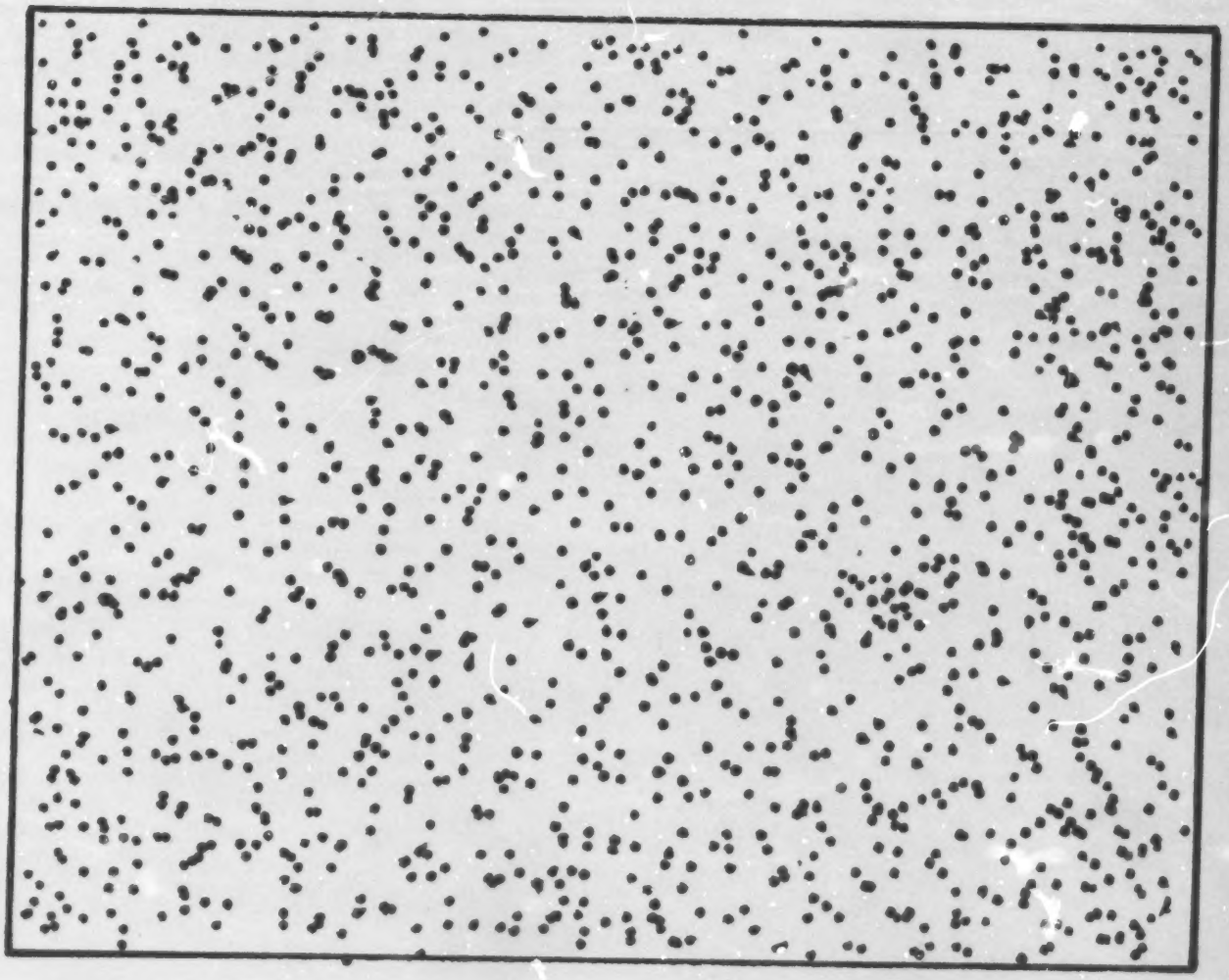
n = 4

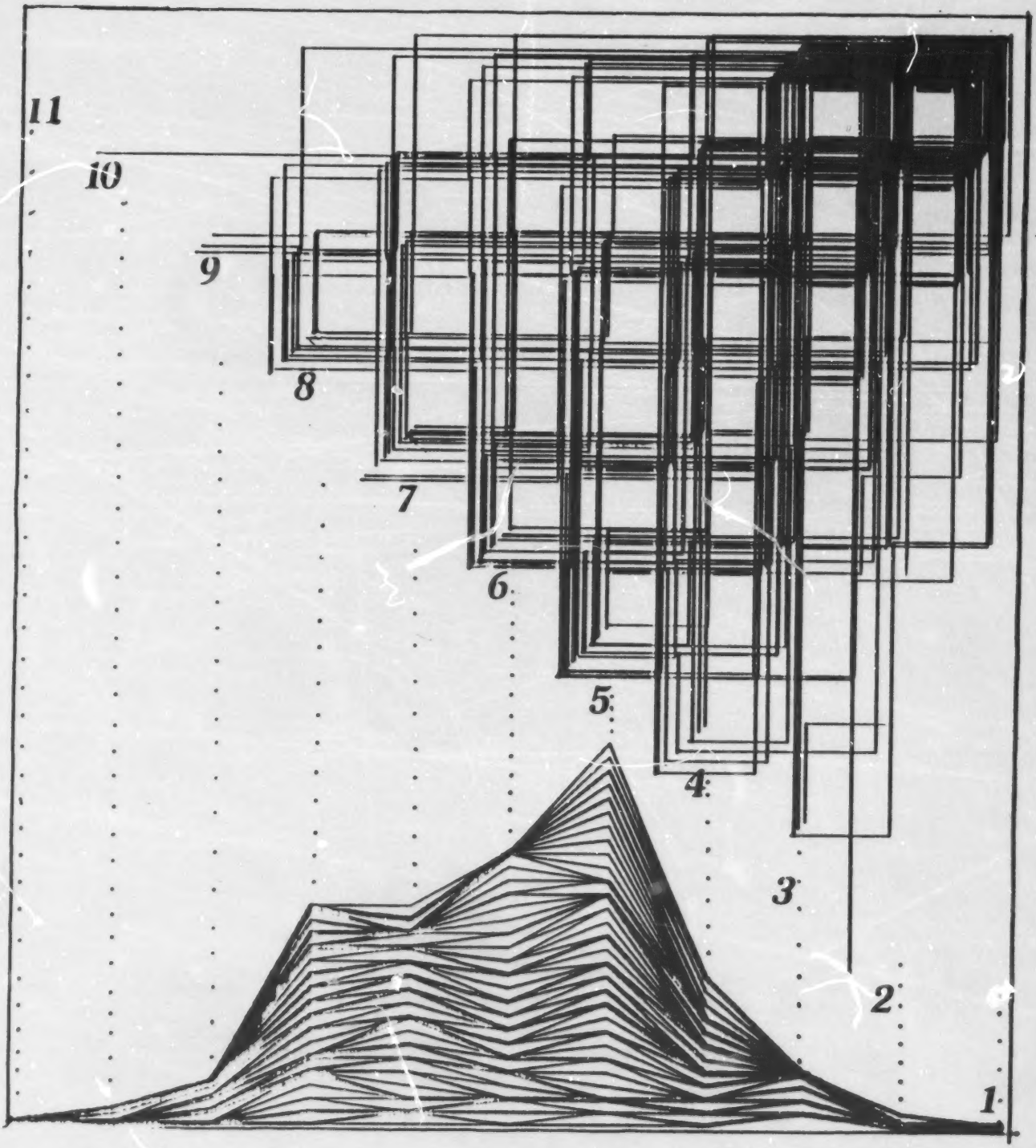
```
10 REM * SIMULATION OF THROWING A N-FACES POLYHEDRON *
20 INPUT "NUMBER OF FACES=";N
30 FOR I = 1 TO 30
40    LET X = RND(N)
50    PRINT X
60 NEXT I
```

n = 6



n = 2

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